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The optimal choice of control constraints☆

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ABSTRACT

The problem of the optimal choice of the limits of a set of possible values of the control during motion for the purpose of obtaining the required form of the attainability set of a linear dynamical system in a specified time interval is considered. Using the method, in which these sets are approximated by ellipsoids, the problem of controlling the parameters of the ellipsoid containing the control vector is solved. Then a functional, which depends on the matrix of the ellipsoid, containing the phase vector, reaches its maximum. The order in which the corresponding formulae are used is illustrated using the example of a simple mechanical system. The results obtained are suitable for systems in which, instead of the control vector, there is an interference vector with controllable boundaries of possible changes and can be extended to stochastic systems.

1. Introduction

We will investigate a linear dynamical system of the form

$$\dot{x} = A(t)x + B(t)u(t) + f(t), \quad x \in R^n, \quad u(t) \in R^m \tag{1.1}$$

All the functions of time considered are such that the solutions of the differential equations, in which these functions are used, exist. All cases in which additional constraints are imposed will be indicated separately.

We explain the meaning of the notation used above. The vector $f(t)$ of dimension n, and also the matrices $A(t)$ of dimension $n \times n$ and $B(t)$ of dimension $n \times m$ are assumed to be specified. Each sample of the phase vector $x(t)$ depends on the corresponding sample of the vector $u(t)$ – control with constraints known in advance. These constraints must be chosen by a method that is optimal is the sense of a certain criterion. This kind of problem can arise in technology, when specifying the powers of control forces along different phase coordinates. The corresponding choice must be made as it applies to the whole ensemble of admissible motions of the system, which requires an investigation of the attainability sets. The construction of multidimensional regions of attainability encounters numerous technical obstacles, that is why, one can use guaranteed estimation to investigate the whole set of possible phase trajectories. We will use the well-known method of external estimates of these regions using ellipsoids (see Refs. 1-3)

$$E(\chi, D) = \left\{y \in R^n; \left(D^{-1}(y-\chi), y-\chi\right) \le 1\right\} \tag{1.2}$$

where χ is the centre and D is a matrix. We will assume that at the initial instant of time $t_0 = 0$, an ellipsoid of dimension $n \times n$ with matrix Q_0 and centre a_0 limits the set of possible values of the vector $x(0)$, while the ellipsoid with matrix $G(t)$ of dimension $m \times m$ and centre at the origin of coordinates contains all admissible values of the vector $u(t)$.

We can then obtain the ellipsoid (1.2) with matrix $Q(t)$, containing the whole set of possible values of $x(t)$ and subject to the equations

$$\dot{Q} = AQ + QA^T + \frac{Q}{q(t)} + q(t)BGB^T, \quad Q(0) = Q_0 \tag{1.3}$$

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where $q(t) > 0$ when $t > 0$. The motion of the centre $a(t)$ of the ellipsoid $Q(t)$ is described by the equation

$$\dot{a} = Aa + f, \quad a(0) = a_0 \tag{1.4}$$

It is of no particular interest in our further discussions.

The scalar function $q(t)$ can be chosen in different ways. It is only important that differential equation (1.3) should have a solution. In particular,[4] we can introduce a functional, smoothly and monotonically dependent on Q, and require that either it reach a minimum at a specified instant of time, or it should have the minimum possible rate of increase over the whole time interval considered. It was suggested in Ref. 5 that one can choose $q(t) = t$. Then, making the replacement $Q(t) = tZ(t)$ in Eq. (1.3) in the case when $Q_0 = 0$, we obtain the following problem, linear in Z:

$$\dot{Z} = AZ + ZA^T + BGB^T, \quad Z(0) = 0 \tag{1.5}$$

This method can also be used to construct the estimating ellipsoids when $Q_0 \neq 0$, since the required attainability set when $Q_0 \neq 0$ is contained in the sum of two ellipsoids: the ellipsoid with matrix $Q_A(t)$ and centre $a(t)$, described by Eq. (1.4), and the ellipsoid with matrix $tZ(t)$ and centre at the origin of coordinates,[5] where the matrix $Q_A(t)$ can be found by solving the Cauchy problem

$$\dot{Q}_A = AQ_A + Q_A A^T, \quad Q_A(0) = Q_0 \tag{1.6}$$

If necessary, this sum can be approximated by a single ellipsoid using well-known methods.[6] Since linear equation (1.6) is independent of the matrix G and its solution presents no particular difficulties we will always assume that $Q_0 = 0$.

Note that Eq. (1.5) is not only convenient by virtue of its linearity, but, in addition, in the majority of cases for small t it describes the true attainability set quite well.[5]

2. Formulation of the problem

In technical applications it may be required to distribute existing control resources when constructing a system in such a way that the attainability set has the maximum possible dimensions in certain directions in phase space. This enables one to improve the controllability when a priori data on the operating conditions of an object are available. The considerations presented in the previous section enable us, for any admissible sample of $u(t)$, to maximize the following functional by choosing the matrix $G(t)$

$$L(Q) = \operatorname{tr}(WQ(T)) + \int_0^T \left(\operatorname{tr}(F(t)Q(t)) - \frac{1}{2}\operatorname{tr}(G(t)R(t)G(t))\right)dt \tag{2.1}$$

where $F(t)$ and the constant W are arbitrary positive-definite symmetric matrices, while T is a finite instant of time. For example, if W and F are the same and diagonal, the choice of the corresponding elements of these matrices determines to what extent, for a given practical application, the value of the maximum of the projection of the attainability set onto any axis of the system of coordinates is important compared with similar projections onto other axes. The choice of the matrix $R(t)$ enables us, as in the linear-quadratic problem in Ref. 7, to limit the norm of the matrix $G(t)$ and, from considerations of, for example, technical applications, we can reflect the limitations on the total maximum control force in system (1.1). From this point of view, the problem consists of choosing the power of each of several drives in order to limit the total weight of motors, their dimensions, the energy consumption level, etc. Unlike the linear-quadratic problem, in this case it is not sufficient to require that the matrix $R(t)$ should only be symmetrical and positive definite. The additional condition will be obtained later.

Taking into account the replacement $Q(t) = tZ(t)$, we will write functional (2.1) in the form

$$L(Z) = \operatorname{tr}(CZ(T)) + \int_0^T \left(\operatorname{tr}(F(t)Z(t))t - \frac{1}{2}\operatorname{tr}(G(t)R(t)G(t))\right)dt \tag{2.2}$$

where $C = TW$ is a constant matrix.

We obtain the following optimal control problem: it is required to find the maximum of functional (2.2) for system (1.5) by choosing the matrix $G(t)$.

3. Solution of the problem

By Pontryagin's maximum principle,[8] for non-autonomous systems, Hamilton's function, as it applies to this problem with a fixed time and free right-hand end of the trajectory, has the form (see, for example, Ref. 7)

$$H = \operatorname{tr}\left(AZP^T\right) + \operatorname{tr}\left(ZA^TP^T\right) + \operatorname{tr}\left(BGB^TP^T\right) + \operatorname{tr}(FZ)t - \frac{1}{2}\operatorname{tr}(GRG)$$

where $P(t)$ is a matrix of conjugate-variable. Its elements can be found as the solution of the following Cauchy problem

$$\dot{P} = -\frac{\partial H}{\partial Z}, \quad P(T) = \frac{\partial \operatorname{tr}(CZ(T))}{\partial Z(T)}$$

Since F and C are symmetrical matrices, we obtain

$$\dot{P} = -A^TP - PA - tF, \quad P(T) = C \tag{3.1}$$

Making the replacement of time $\tau = T - t$ in (3.1), we obtain

$$\frac{dP}{d\tau} = \Psi P + P\Psi^T + (T-\tau)F, \quad P(0) = C, \quad \tau \in [0;T]; \quad \Psi = A^T \tag{3.2}$$

System (3.2) can be regarded as a special case of system (1.5) apart from the replacement of the symmetrical non-negative definite matrix BGB^T by the symmetrical non-negative definite matrix $(T-\tau)F$. Consequently, according to well-known results,[3] $P(t)$ is a symmetrical non-negative definite matrix.

In order to obtain the extremum of H as a function of G, we will use the derivative

$$\frac{\partial H}{\partial G} = B^T PB - GR \tag{3.3}$$

As shown by Formal'skii[7] for a more general case, the symmetrical $m^2 \times m^2$ matrix $-\partial^2 H/\partial G^2$ will be positive definite by virtue of the fact that the matrix R is positive definite. Since the matrix R is non-degenerate, we obtain from (3.3) the unique required optimum matrix, which gives a maximum of the function H:

$$G^*(t) = B^T PBR^{-1} \tag{3.4}$$

Note that this matrix must be symmetrical. Consequently, the symmetrical matrices $B^T PB$ and R^{-1} must be interchangeable, i.e., the matrix $R^{-1}(t)$ must be a solution of the corresponding Frobenius problem.[9] For the purposes of this paper, it is sufficient to ensure that the norm of the matrix $G^*(t)$ should be bounded, and hence we can put

$$R(t) = \rho(t)I \tag{3.5}$$

where $\rho(t)$ is an arbitrary scalar positive function, while I is the identity matrix.

We will show that the matrix G^* is positive definite. Using the well-known approach,[9] we will introduce the symmetrical matrix $Y = 1/\sqrt{R}$. Then, equality (3.4) takes the form $G^* = Y^{-1}SY$, where $S = YB^T PBY$ is a symmetrical matrix. It follows from the fact that the matrix G^* is similar to the symmetrical matrix S that it has a simple structure and real characteristic numbers $\lambda_1, \ldots \lambda_m$. They can be found from the equation $\det(B^T PB - \lambda R) = 0$, which, by virtue of the fact that the matrix $B^T PB$ is non-negative definite and the fact that the matrix R is positive definite, has non-negative roots.

Thus, for the above-mentioned conditions, a unique optimal symmetrical positive definite solution G^* always exists. It can be found from formula (3.4), the values of $P(T)$ in which can be obtained by solving Cauchy problem (3.1).

4. Example

Consider the motion of two masses with coordinates x_1 and x_2, connected by a spring of stiffness k, along a straight line under the control forces F_1 and F_2, where the first of these acts on the first mass and the second acts on the second mass. The equations of this mechanical system have the form

$$m_1\ddot{x}_1 = F_1 + k(x_2 - x_1), \quad m_2\ddot{x}_2 = F_2 + k(x_1 - x_2) \tag{4.1}$$

After converting relations (4.1) to normal form, the coordinate and velocity of the first mass will correspond to the first and second variables in phase space, while the coordinate and velocity of the second mass will correspond to the third and fourth variables. In the notation of (1.1) we obtain

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & 0 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & 0 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{Vmatrix}, \quad f = \mathrm{col}(0,0,0,0)$$

We will assume that at the initial instant of time $t = 0$, the values of all the phase variables are known exactly and are equal to zero. Then, by Eq. (1.4), the vector $a(t)$ of the centre of the estimate is identically equal to zero throughout the whole time of the process T.

Suppose that, at any instant of time, the control vector with components F_1 and F_2 belongs to ellipsoid (1.2) with centre at the origin of coordinates, and matrix $G(t)$, which must be chosen so that the value of functional (2.2) (and thereby (2.1) also reaches a maximum. We will choose the matrices W and F in (2.1) to be identity matrixes. In other words, we must ensure that the trace of the matrix $Q(t)$ is as large as possible. We will let the matrix R be identity matrix.

The required matrix will have the form

$$G^*(t) = \begin{Vmatrix} \dfrac{P_{22}(t)}{m_1^2} & \dfrac{P_{24}(t)}{m_1 m_2} \\ \dfrac{P_{24}(t)}{m_1 m_2} & \dfrac{P_{44}(t)}{m_2^2} \end{Vmatrix}$$

Despite the fact that Cauchy problem (3.1) was solved analytically in this case, because of the unwieldy nature of the formulae we will confine ourselves to a graphic representation of the results for values of the parameters $k = 10^3$ N/m, $m_1 = 1$ kg and $m_2 = 2$ kg. The process time T was chosen to be one second. In Fig. 1 we show graphs, in SI units, of the functions $G^*_{11}(t)$, $G^*_{22}(t)$ and $G^*_{12}(t)$, denoted by the numbers 1, 2 and 3. It is clear that the relations between the values of the elements of the matrix $G^*(t)$ primarily have practical meaning at each

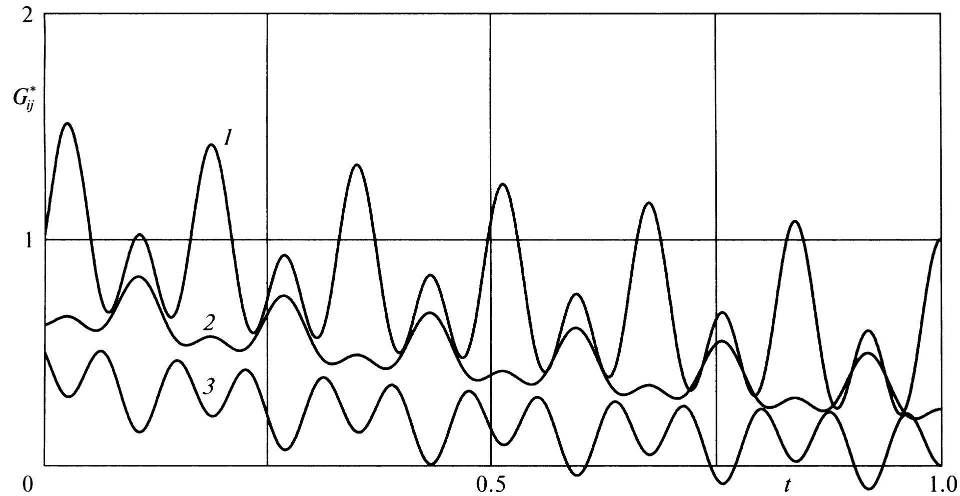

Fig. 1.

instant of time, since the norm of this matrix depends on the choice of the function $\rho(t)$ in equality (3.5). In particular, putting the function $\rho(t)$ equal to the trace $B(t)^{\mathrm{T}}P(t)B(t)$, we could achieve equality of the trace $G^*(t)$ to unity.

5. Supplements

The solution obtained allows of a number of generalizations.

1. The above problem was investigated for the case when in system (1.1) for $x(0)\in E(Q_0,a_0)$ and $u(t)\in E(G,0)$ the matrix B is a known function of time while G is an unknown function. We will assume that $B=B(t,\xi)$ and $G=G(t,\xi)$, where $\xi=\xi(t)$ is an unknown parameters vector of dimension k. If the following identity holds

$$B(t,\xi)G(t,\xi)B^T(t,\xi) \equiv B_1(t)G_1(\xi)B_1^T(t)$$

   where the matrix G_1 can be taken as arbitrary, the solution obtained can be used after replacing B and G by B_1 and G_1 respectively. In particular, if $u(t)\in E(I,0)$ and we must choose the elements of B, then B_1 = I while $G_1 = BB^T$. At the concluding stage, to determine the values of the elements of the optimal matrix B^*, it is necessary to solve the equation $B^*B^{*^T} = G_1^*$, where G_1^* is the optimal matrix, similar to G^*.
2. If $u(t)$ is not a control, but some arbitrary perturbation, having assigned boundaries, the solution obtained enables the attainability set of system (1.1) to be controlled, independently of the realization of the perturbation within these boundaries.
3. As follows from the results obtained previously in Ref. 10, a method of constructing a system with limited perturbations exists, based on data forming the basis of a system with "white noise" type perturbations. From this point of view the problem which has been solved above enables one to choose the corresponding parameters in an optimal manner using stochastic estimation.

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References

1. Schweppe FC. Recursive state estimation: unknown but bounded errors and system inputs. *IEEE Trans Automat Control* 1968;**V.AC-13**(No. 1):22–8.
2. Kurzhanskii AB. *Control and Observation under Conditions of Uncertainty*. Moscow: Nauka; 1977.
3. Chernousko FL. *State Estimation for Dynamic Systems*. Boca Raton: CRC Press; 1994.
4. Ovseyevich AI, Taraban'ko Yu V. Explicit formulae for ellipsoids, which approximate regions of attainability. *Izv Ross Akad Nauk Teor Sistemy Upravl* 2007;**2**:33–44.
5. Shmatkov AM. Non-singular locally optimal ellipsoidal approximation of the estimate of the states of linear systems. *Prikl Mat Mekh* 2008;**72**(2):241–50.
6. Reshetnyak YuN. Summation of ellipsoids in the problem of guaranteed estimation. *Prikl Mat Mekh* 1989;**53**(2):249–54.
7. Roitenberg Ya N. *Automatic Control*. Moscow: Nauka; 1992.
8. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mishchenko Ye F. *The Mathematical Theory of Optimal Processes*. New York: Wiley; 1962.
9. Gantmacher FR. *The Theory of Matrices*. New York: Chelsea; 1974.
10. Shmatkov AM. Comparison of stochastic and ellipsoidal estimation of uncertainty for a dynamical system with perturbations, bounded in value. *Dokl Ross Akad Nauk* 2006;**411**(4):460–3.

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